1. A circle C with centre at $(-2, 6)$ passes through the point $(10, 11)$.

(a) Show that the circle C also passes through the point $(10, 1)$.

(3)

The tangent to the circle C at the point $(10, 11)$ meets the y-axis at the point P and the tangent to the circle C at the point $(10, 1)$ meets the y axis at the point Q.

(b) Show that the distance PQ is 58 explaining your method clearly.

(7)

(a) Radius of circle = distance between $(-2,6)$ and $(10,11)$

$$\text{Radius} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{(-2-10)^2+(6-11)^2}$$

$$= \sqrt{(-12)^2+(-5)^2}$$

$$= \sqrt{144+25}$$

$$= \sqrt{169}$$

$$= 13 \text{ units}$$

Now, distance between $(-2,6)$ and $(10,1)$:

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{(-2-10)^2+(6-1)^2}$$

$$= \sqrt{(-12)^2+5^2}$$

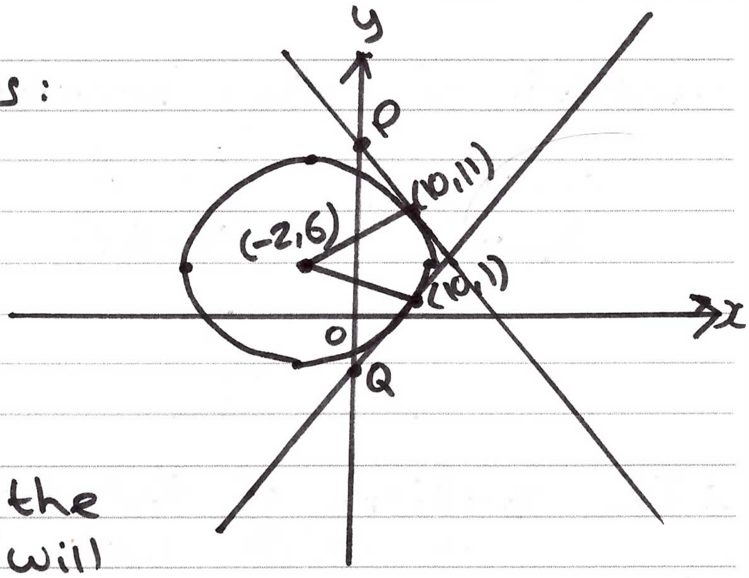
$$= \sqrt{144+25}$$

$$= \sqrt{169}$$

$$= 13 \text{ units}$$

Distance are equal, and so $(10,1)$ lies on the circle

Question continued



(b) Gradient of radius:

$$\frac{y_2-y_1}{x_2-x_1} = \frac{11-6}{10-(-2)}$$

$$= \frac{5}{12}$$

$\therefore$ the gradient of the tangent at (10,11) will be $-\frac{12}{5}$

Equation of tangent at (10,11):

$$y-y_1 = m(x-x_1)$$

$$y-11 = -\frac{12}{5}(x-10)$$

$$5(y-11) = -12(x-10)$$

$$5y-55 = -12x+120$$

$$12x+5y-175=0$$

When this line cuts the y-axis, $x=0$

$$\therefore \quad 5y-175=0$$

$$5y=175 \Rightarrow y=35$$

$\therefore$ P is at $(0,35)$

Question continued

Gradient of radius between centre and (10,1):

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 6}{10 - (-2)}$$

$$= \frac{-5}{12}$$

$\therefore$ the gradient of the tangent at (10,1) will be $\frac{12}{5}$.

Equation of tangent at (10,1):

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{12}{5}(x - 10)$$

$$5(y-1) = 12(x-10)$$

$$5y - 5 = 12x - 120$$

$$12x - 5y - 115 = 0$$

When this line cuts the y-axis, $x = 0$

$\therefore -5y = 115 \Rightarrow y = -23$

$\therefore$ Q is at $(0, -23)$

Distance PQ $= 35 + 23 \boxed{= 58}$

(Total for Question is 10 marks)

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2. The circle C has equation

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

(a) Find

(i) the coordinates of the centre of C

(ii) the radius of C

(3)

The line with equation $y = kx$, where k is a constant, cuts C at two distinct points.

(b) Find the range of values for k.

(6)

ai) $(x-3)^2 + (y+5)^2 - 9 - 25 + 9 = 0$

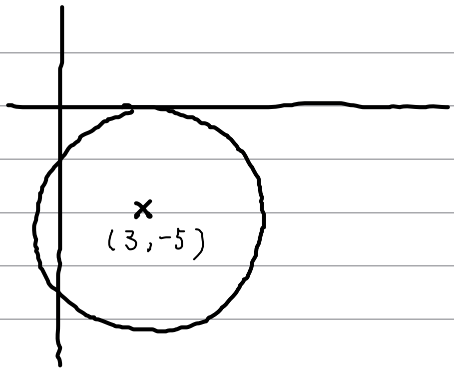
$(x-3)^2 + (y+5)^2 = 25$

centre : $(3, -5)$

ii) $r = \sqrt{25}$

$= 5$ cm

b)



$y = kx$

$x^2 + (kx)^2 - 6x + 10(kx) + 9 = 0$

$(1+k^2)x^2 + (10k-6)x + 9 = 0$

$a = 1+k^2 \quad b = 10k-6 \quad c = 9$

$b^2 - 4ac > 0$

$(10k-6)^2 - 4(1+k^2)(9) = 100k^2 - 120k + 36 - 36 - 36k^2$

$= 64k^2 - 120k$

$= k(8k-15)$

C.V. $\quad k(8k-15) = 0$

$k = 0 \quad k = \frac{15}{8}$

$b^2 - 4ac > 0$ so $\quad k < 0 \; , \; k > \frac{15}{8}$

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3. A circle C has equation

$$x^2 + y^2 - 4x + 8y - 8 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The straight line with equation $x = k$, where k is a constant, is a tangent to C.

(b) Find the possible values for k.

(2)

a) $x^2 - 4x + y^2 + 8y = 8$

$(x-2)^2 - 4 + (y+4)^2 - 16 = 8$

$(x-2)^2 + (y+4)^2 = 28$ //

so centre $(2, -4)$

radius $\sqrt{28} = 2\sqrt{7}$

b) $\underline{x=4}$: $(k-2)^2 + (y^2+8y+16) = 28$

$y^2 + 8y + ((k-2)^2 - 12) = 0$

$x=k$ is a tangent ; ie there is only one intersection.

so $b^2 - 4ac = 0$

$(8)^2 - 4(1)((k-2)^2 - 12) = 0$

$16 = (k-2)^2 - 12$

$28 = (k-2)^2$

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Question continued

$\therefore \quad k - 2 = \pm 2\sqrt{7}$

$\boxed{k = 2 \pm 2\sqrt{7}}$

(Total for Question is 5 marks)

4. (i) A circle C_1 has equation

$$x^2 + y^2 + 18x - 2y + 30 = 0$$

The line l is the tangent to C_1 at the point $P(-5, 7)$.

Find an equation of l in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(5)

(ii) A different circle C_2 has equation

$$x^2 + y^2 - 8x + 12y + k = 0$$

where k is a constant.

Given that C_2 lies entirely in the 4th quadrant, find the range of possible values for k.

(4)

i. complete the square to find centre of circle

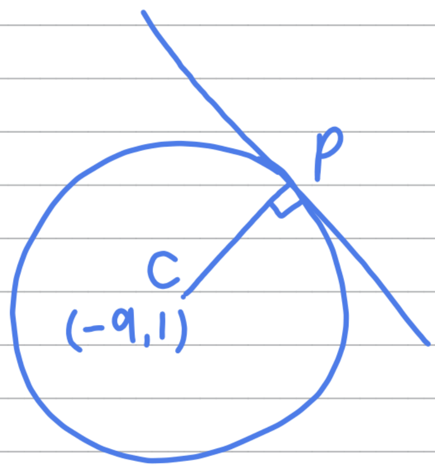
$x^2 + y^2 + 18x - 2y + 30 = 0$

$\hookrightarrow (x+9)^2 - 81 + (y-1)^2 - 1 + 30 = 0$

$\therefore$ centre $(-9, 1)$ (don't need radius)

we can use the fact that the radius & tangent are ⊥ to find gradient of the tangent : $m_r \times m_l = -1$

gradient of radius joining C to P :



$\frac{7-1}{-5+9} = \frac{3}{2}$

$\therefore m_l = -\frac{2}{3}$

using $y - y_0 = m(x - x_0)$: $y - 7 = -\frac{2}{3}(x+5)$

$3y - 21 = -2x - 10$

DO NOT WRITE IN THIS AREA

Question continued

$\therefore l: 2x+3y-11=0$

ii. lies in 4^{th} quadrant $\Rightarrow$ need centre of C_2

$x^2+y^2-8x+12y+k=0$

$\hookrightarrow (x-4)^2-16+(y+6)^2-36+k=0$

$\Rightarrow (x-4)^2+(y+6)^2=52-k$

centre $(4,-6)$

to lie entirely in one quadrant, can't cross axes

$\Rightarrow$ radius must be less than shortest distance from axes

$\therefore r<4 \Rightarrow 52-k<4^2$

$\therefore k>36$

lengths can't be negative

$r>0 \Rightarrow 52-k>0$

so in total, $\underline{36<k<52}$

5.

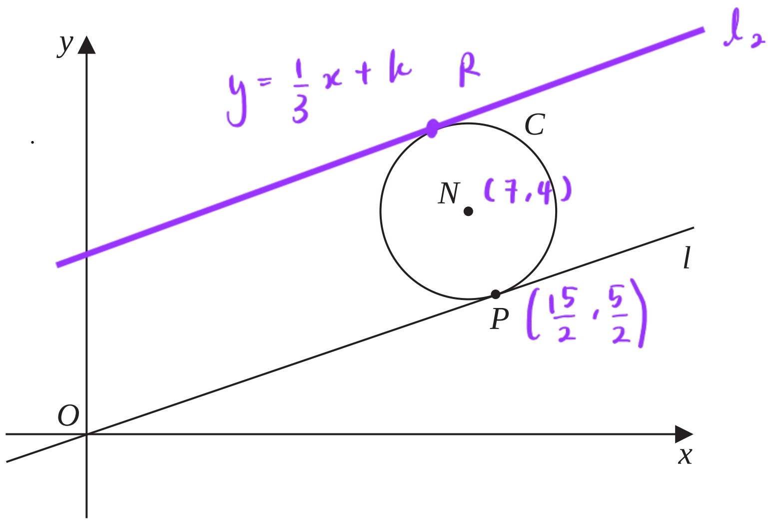


Figure 4

Figure 4 shows a sketch of a circle C with centre $N(7, 4)$

The line l with equation $y = \frac{1}{3}x$ is a tangent to C at the point P.

Find

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants,

(2)

(b) an equation for C.

(4)

The line with equation $y = \frac{1}{3}x + k$, where k is a non-zero constant, is also a tangent to C.

(c) Find the value of k.

(3)

(a) line l has equation $y = \frac{1}{3}x$. Hence, the gradient is $\frac{1}{3}$

gradient of PN $= \frac{-1}{1/3} = -3$

Use coordinates of N (7, 4) to form the equation.

PN : $y - (4) = -3(x - 7)$ 1

PN : $y - 4 = -3x + 21$

PN : $y = -3x + 25$ \# 1

Question 5 continued

(b) Find coordinates of P using line l and PN

$$\frac{1}{3}x = -3x + 25 \quad (1)$$

$$\frac{1}{3}x + 3x = 25$$

$$\frac{10}{3}x = 25$$

$$x = \frac{25 \times 3}{10}$$

$$x = 7.5$$

$$x = \frac{15}{2}$$

$y = \frac{1}{3} \times \frac{15}{2}$ → substitute x into $y = \frac{1}{3}x$ to find the y coordinate

$$= \frac{15}{6}$$

$$= \frac{5}{2}$$

$$\therefore P\left(\frac{15}{2}, \frac{5}{2}\right) \quad (1)$$

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$r^2 = \left(\frac{15}{2} - 7\right)^2 + \left(\frac{5}{2} - 4\right)^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2$$

$$= \frac{1}{4} + \frac{9}{4}$$

$$= \frac{10}{4}$$

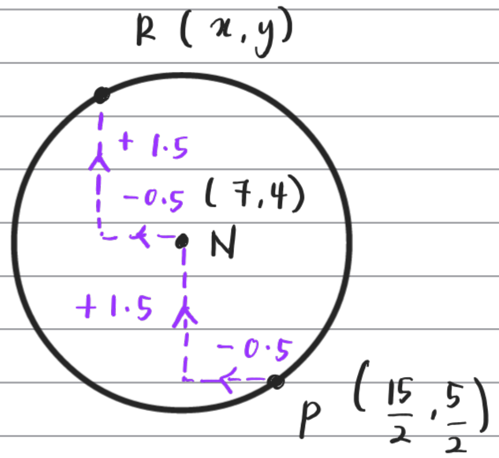
$$r^2 = \frac{5}{2}$$

$$r = \sqrt{\frac{5}{2}} \quad (1)$$

Question 5 continued

equation for C : $(x-7)^2 + (y-4)^2 = \frac{5}{2}$ # (1)

(c)



Coordinates of R: $((7-0.5), (4+1.5))$

R: $(6.5, 5.5)$

R: $\left(\frac{13}{2}, \frac{11}{2}\right)$ (1)

Given $y = \frac{1}{3}x + k$ substitute coordinate of R into this

$\frac{11}{2} = \frac{1}{3}\left(\frac{13}{2}\right) + k$ (1)

$\frac{11}{2} = \frac{13}{6} + k$

$k = \frac{11}{2} - \frac{13}{6}$

$k = \frac{10}{3}$ # (1)

6. A circle C has equation

$$x^2 + y^2 - 4x + 10y = k$$

where k is a constant.

(a) Find the coordinates of the centre of C.

(2)

(b) State the range of possible values for k.

(2)

a) $x^2 + y^2 - 4x + 10y = k$

$\underbrace{x^2 - 4x} + \underbrace{y^2 + 10y} = k$

$(x-2)^2 - 4 + (y+5)^2 - 25 = k$ 1

$x : 2$ and $y : -5$ $\Rightarrow$ $C : \underline{(2, -5)}$ 1

b) What do we know about the radius? $r > 0$

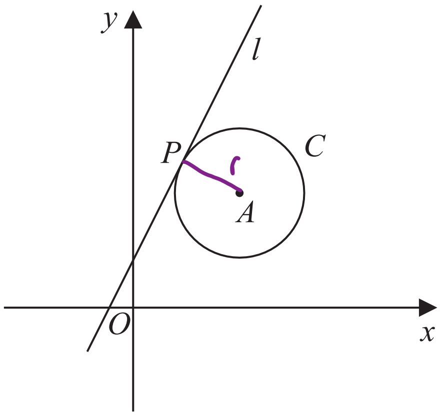
$(x-2)^2 + (y+5)^2 - \cancel{29} = k + 29$ $\Rightarrow$ $k + 29 > 0$

$\Rightarrow k > -29$

$\Rightarrow$ $\underline{k > -29}$ 1

(Total for Question is 4 marks)

7.



Not to scale

r = radius of c

Figure 3

The circle C has centre A with coordinates $(7, 5)$.

The line l, with equation $y = 2x + 1$, is the tangent to C at the point P, as shown in Figure 3.

↳ $m_t = 2$

(a) Show that an equation of the line PA is $2y + x = 17$

(3)

(b) Find an equation for C.

(4)

The line with equation $y = 2x + k$, $k \neq 1$ is also a tangent to C.

(c) Find the value of the constant k.

(3)

a) m_t = tangent Gradient. m_r = radius gradient.

for perpendicular lines, $m_1 m_2 = -1$

$m_t \times m_r = -1$

$2 \times m_r = -1$

$m_r = -\frac{1}{2}$

$y - y_1 = m(x - x_1)$

(x_1, y_1) is a point on the line

$x_1 = 7 \quad y_1 = 5$

$y - 5 = -\frac{1}{2}(x - 7)$

$2y - 10 = -(x - 7)$

$2y - 10 = -x + 7$

$2y + x = 17$ as required.

Question continued

b)

$PA: 2y + x = 17 \quad l: y = 2x+1 \quad A(7,5)$

$(x-a)^2 + (y-b)^2 = r^2$

$(x-7)^2 + (y-5)^2 = r^2$

$2(2x+1) + x = 17$

$4x + 2 + x = 17$

$5x + 2 = 17$ ✓

$5x = 15 \quad \therefore x = 3 \quad \Rightarrow y = 2(3)+1$

$= 6+1$

$= 7.$

$P = (3, 7)$ ✓

$|PA| = \sqrt{(P_x - A_x)^2 + (P_y - A_y)^2}$

$= \sqrt{(3-7)^2 + (7-5)^2} = \sqrt{16+4} = \sqrt{20}$ ✓

$r = \sqrt{20} \quad \therefore r^2 = 20$

Equation of C is $(x-7)^2 + (y-5)^2 = 20$ ✓

Question continued

c)

$C: (x-7)^2 + (y-5)^2 = 20 \qquad y = 2x + k$

tangent $\Rightarrow$ solution exist.

$C: x^2 - 14x + 49 + y^2 - 10y + 25 = 20$

$x^2 - 14x + y^2 - 10y + 54 = 0$

$x^2 - 14x + (2x+k)^2 - 10(2x+k) + 54 = 0$

$x^2 - 14x + 4x^2 + 4kx + k^2 - 20x - 10k + 54 = 0$

$5x^2 + (4k-34)x + k^2 - 10k + 54 = 0$

$ax^2 + bx + c$

tangent $\Rightarrow$ one solution only : $b^2 - 4ac = 0$

$(4k-34)^2 - \underbrace{4(5)}_{-20}(k^2 - 10k + 54) = 0$

$16k^2 - 272k + 1156 - 20k^2 + 200k - 1080 = 0$

$-4k^2 - 72k + 76 = 0$

$k^2 + 18k - 19 = 0$

$(k+19)(k-1) = 0 \qquad k + 19 = 0 \Rightarrow k = -19$

$\qquad\qquad\qquad\qquad k - 1 = 0 \Rightarrow k = 1$

$k = -19$ or 1, but since $k \neq 1$, $\therefore k = -19$

8.

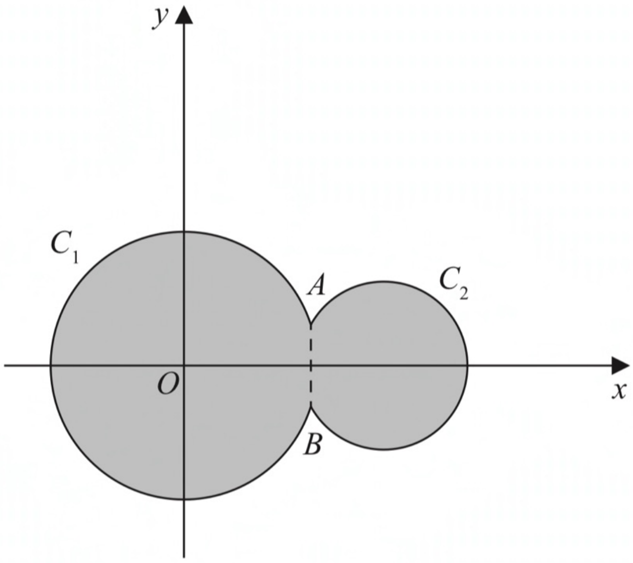


Figure 3

Circle C_1 has equation $x^2 + y^2 = 100$

Circle C_2 has equation $(x - 15)^2 + y^2 = 40$

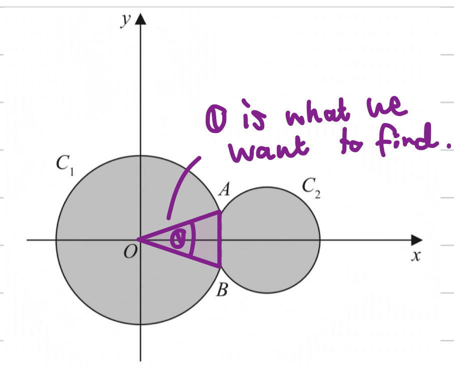
The circles meet at points A and B as shown in Figure 3.

(a) Show that angle $AOB = 0.635$ radians to 3 significant figures, where O is the origin.

(4)

a) $C_1 : x^2 + y^2 = 100$ and $C_2 : (x-15)^2 + y^2 = 40$

$y^2 = 100 - x^2$ (substitute this into C_2)



$\Rightarrow (x-15)^2 + 100 - x^2 = 40$

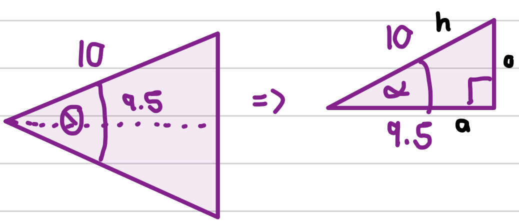
$x^2 - 30x + 225 + 100 - x^2 = 40$

$30x = 285$

$x = \frac{285}{30} = \frac{19}{2}$, or $x = 9.5$. (1) Then $y^2 = 100 - (9.5)^2$

$y^2 = \frac{39}{4} \Rightarrow y = \pm \frac{\sqrt{39}}{2}$

$\Rightarrow A : (9.5, 3.12)$ and $B : (9.5, -3.12)$ $\Rightarrow y = \pm 3.12$ (1)



let $\alpha = \frac{\theta}{2}$ then α : $\cos\alpha = \left(\frac{9.5}{10}\right)$

$\alpha = \cos^{-1}(9.5/10)$

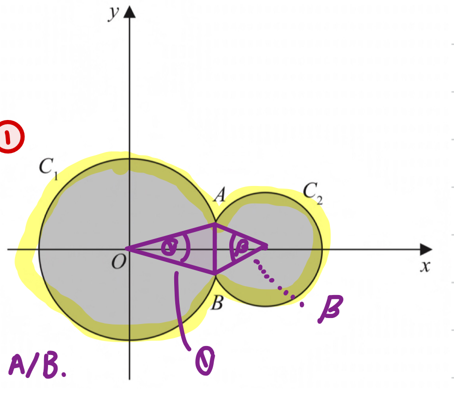
$\alpha = 0.31756$ (1)

$\Rightarrow \theta = 2\alpha = 2 \times 0.31756 = 0.63512 \Rightarrow$ The angle $AOB = 0.635$ as required. (1)

The region shown shaded in Figure 3 is bounded by C_1 and C_2

(b) Find the perimeter of the shaded region, giving your answer to one decimal place.

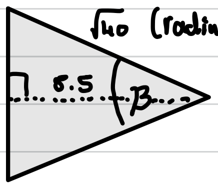
(4)



b) For C_1, we know that $\theta = 0.635$ radians (from part a) and we also know that the radius is 10.

$\Rightarrow$ Perimeter of C_1 (P_1) ; $P_1 = 10 \times (2\pi - 0.635) = \underline{56.48}$ ①

For C_2 :



Centre of C_2

... line : $15 - 9.5 = 5.5$

x coordinate of A/B.

$\beta = 2 \times \cos^{-1}\left(\frac{5.5}{\sqrt{40}}\right) \Rightarrow \beta = 1.03$ radians. ①

$\Rightarrow$ Perimeter of C_2, (P_2) ; P_2 : $\sqrt{40} \times (2\pi - 1.03) = 33.22.$ ①

$\Rightarrow$ Total Perimeter $= P_1 + P_2 = 56.48 + 33.22$

$\Rightarrow$ Total Perimeter $= \underline{89.7}.$ ①

9. A circle C with radius r

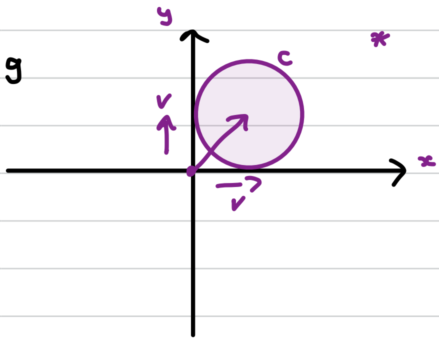
- lies only in the 1st quadrant
- touches the x-axis and touches the y-axis

The line l has equation $2x + y = 12$

(a) Show that the x coordinates of the points of intersection of l with C satisfy

$$5x^2 + (2r - 48)x + (r^2 - 24r + 144) = 0$$

(3)



• The centre of the circle is shifted by r units along the x and the y axis.

$2x + y = 12$

$\Rightarrow$ $C: \ (x-r)^2 + (y-r)^2 = r^2$ $\quad l: \ y = 12 - 2x$

$\Rightarrow$ $x^2 - 2rx + r^2 + y^2 - 2ry + r^2 = r^2$

$\Rightarrow$ $x^2 + y^2 - 2rx - 2ry + r^2 = 0$ 1 $\quad$ Substitute this in!

$\Rightarrow$ $x^2 + (12-2x)^2 - 2rx - 2r(12-2x) + r^2 = 0$

1

$\Rightarrow$ $x^2 + 144 - 48x + 4x^2 - 2rx - 24r + 4rx + r^2 = 0$

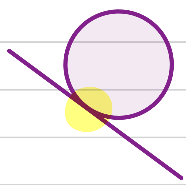
$\Rightarrow$ $5x^2 - 48x + 2rx + (r^2 - 24r + 144) = 0$

$\Rightarrow$ $5x^2 + (2r-48)x + (r^2 - 24r + 144) = 0$ as required. 1

Given also that l is a tangent to C,

(b) find the two possible values of r, giving your answers as fully simplified surds.

(4)



discriminant

Tangent $\Rightarrow$ $b^2 - 4ac = 0$ Since one repeated root.

Recall from part a we have that $5x^2 + (2r-48)x + (r^2 - 24r + 144) = 0$

$a = 5$, $b = 2r - 48$ and $c = r^2 - 24r + 144$

$\Rightarrow$ $b^2 - 4ac = 0$ $\Rightarrow$ $(2r-48)^2 - 4 \times 5(r^2 - 24r + 144) = 0$ 1

$\Rightarrow$ $4r^2 - 192r + 2304 - 20r^2 + 480r - 2880 = 0$

$\Rightarrow$ $-16r^2 + 288r - 576 = 0$

$\div -16$ $\quad \Rightarrow$ $r^2 - 18r + 36 = 0$ 1 $\Rightarrow$ Quadratic formula: $a = 1$ $\quad b = -18$ $\quad c = 36$

$\Rightarrow$ $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{18 \pm \sqrt{(-18)^2 - 4(1)(36)}}{2}$ 1 $= \dfrac{18 \pm 6\sqrt{5}}{2}$ $\Rightarrow$ $r = 9 \pm 3\sqrt{5}$ 1

10. The circle C has equation

$$x^2 + y^2 - 10x + 4y + 11 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C, giving your answer as a simplified surd.

(4)

The line l has equation $y = 3x + k$ where k is a constant.

Given that l is a tangent to C, $(b^2 - 4ac = 0)$

(b) find the possible values of k, giving your answers as simplified surds.

(5)

(a) $(x-5)^2 - (-5)^2 + (y+2)^2 - (2)^2 + 11 = 0$

half of coefficient x (i.e. 10)   half of coefficient y (i.e. 4)

1

$(x-5)^2 + (y+2)^2 = 25 + 4 - 11$

$(x-5)^2 + (y+2)^2 = 18$

(a)(i) $(5, -2)$ 1

(ii) $r = \sqrt{18} = 3\sqrt{2}$ 1 1

(b) $x^2 + y^2 - 10x + 4y + 11 = 0$ — (1)

$y = 3x + k$ — (2)

Substitute (2) into (1)

$x^2 + (3x+k)^2 - 10x + 4(3x+k) + 11 = 0$

$x^2 + 9x^2 + 6kx + k^2 - 10x + 12x + 4k + 11 = 0$ 1

$10x^2 + (6k+2)x + 11 + 4k + k^2 = 0$ 1

Question continued

$b^2 - 4ac = 0$

$(6k+2)^2 - 4(10)(11+4k+k^2) = 0$ (1)

$36k^2 + 24k + 4 - 440 - 160k - 40k^2 = 0$

$-4k^2 - 136k - 436 = 0$

$4k^2 + 136k + 436 = 0$ (1)

$\therefore k = -17 \pm 6\sqrt{5}$ (1)